Journal of Approximation Theory **104**, 220–225 (2000)
doi:10.1006/jath.1999.3448, available online at http://www.idealibrary.com on IDEAL®

On Mean Convergence of Lagrange Interpolation for General Arrays

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Communicated by Paul Nevai

Received May 10, 1999; accepted in revised form November 18, 1999

For $n \geqslant 1$, let $\{x_{jn}\}_{j=1}^{n}$ be n distinct points in a compact set $K \subset \mathbb{R}$ and let $L_n[\cdot]$ denote the corresponding Lagrange interpolation operator. Let v be a suitably restricted function on K. What conditions on the array $\{x_{jn}\}_{1 \leqslant j \leqslant n,\, n \geqslant 1}$ ensure the existence of $p > 0$ such that $\lim_{n \to \infty} \|(f - L_n[f])\, v\|_{L_p(K)} = 0$ for very continuous $f: K \to \mathbb{R}$? We show that it is necessary and sufficient that there exists $r > 0$ with $\sup_{n \geqslant 1} \|\pi_n v\|_{L_r(K)} \sum_{j=1}^{n} (1/|\pi_n'|(x_{jn})) < \infty$. Here for $n \geqslant 1$, π_n is a polynomial of degree n having $\{x_{jn}\}_{j=1}^{n}$ as zeros. The necessity of this condition is due to Ying Guang Shi.

1. THE RESULT

There is a vast literature on mean convergence of Lagrange interpolation, based primarily at zeros of orthogonal polynomials and their close cousins. See [3–10] for recent references. Most of the work dealing with mean convergence of Lagrange interpolation for general arrays involves necessary conditions [6, 9], since sufficient conditions are hard to come by. Some sufficient conditions for convergence of general arrays in L_p, $p > 1$, have been given in [3].

In a recent paper, the author showed that distribution functions and Loomis' Lemma may be used to investigate mean convergence of Lagrange interpolation in L_p, $p < 1$ [2]. Indeed those techniques show that investigating convergence of Lagrange interpolation in L_p is inherently easier for $p < 1$ than for $p \geqslant 1$. Here we show that similar ideas may be used to solve the problem of whether there is convergence in weighted L_p spaces for at least one $p > 0$.

Throughout, we consider an array X of interpolation points $X = \{x_{jn}\}_{1 \leqslant j \leqslant n,\, n \geqslant 1}$ in a compact set $K \subset \mathbb{R}$, with

$$x_{nn} < x_{n-1,n} < \cdots < x_{2n} < x_{1n}.$$

0021-9045/00 $35.00

We denote by $L_n[\cdot]$ the associated Lagrange interpolation operator, so that for $f\colon K \to \mathbb{R}$, we have

$$L_n[f](x) = \sum_{j=1}^{n} f(x_{jn})\, \ell_{jn}(x),$$

where the fundamental polynomials $\{\ell_{kn}\}_{k=1}^{n}$ satisfy

$$\ell_{kn}(x_{jn}) = \delta_{jk}.$$

We also let π_n denote a polynomial of degree n (without any specific normalisation) whose zeros are $\{x_{jn}\}_{j=1}^{n}$. Our result is:

THEOREM 1. *Let $K \subset \mathbb{R}$ be compact, and let $v \in L_q(K)$ for some $q > 0$. Let the array X of interpolation points lie in K. The following are equivalent:*

(I) *There exists $p > 0$ such that for every continuous $f\colon K \to \mathbb{R}$, we have*

$$\lim_{n \to \infty} \|(f - L_n[f])\, v\|_{L_p(K)} = 0. \tag{1}$$

(II) *There exists $r > 0$ such that*

$$\sup_{n \geq 1} \|\pi_n v\|_{L_r(K)} \left(\sum_{j=1}^{n} \frac{1}{|\pi'_n|(x_{jn})} \right) < \infty. \tag{2}$$

Remarks. (a) The new feature is the sufficiency; the necessity is essentially due to Ying Guang Shi [9]. An alternative way to formulate (2) is

$$\sup_{n \geq 1} \|S_n v\|_{L_r(K)} < \infty,$$

where

$$S_n(x) := \sum_{j=1}^{n} |(x - x_{jn})\, \ell_{jn}(x)| = |\pi_n(x)| \sum_{j=1}^{n} \frac{1}{|\pi'_n|(x_{jn})}. \tag{3}$$

Indeed, Shi [9] used this in necessary conditions on $[-1, 1]$.

(b) Note that if (2) holds for a given r, it holds for any smaller r. Likewise if (1) holds for some $p > 0$, then it holds for all smaller p. Our proof shows that if (2) holds for a given r, then (1) holds for $p < \min\{\frac{1}{2}, \frac{r}{2}, q\}$. Conversely if (1) holds for a given p, then (2) holds with $r = p$.

(c) Note that K could, for example, consist of finitely many intervals. What is somewhat restrictive is the formulation of (2). We may insert a weight w in (2), so that it becomes

$$\sup_{n\geqslant 1} \|\pi_n v\|_{L_r(K)} \left(\sum_{j=1}^{n} \frac{1}{|\pi_n' w|(x_{jn})} \right) < \infty.$$

The advantage of this is that the requirement on the $\{x_{jn}\}$ is weakened, if $w(x)$ approaches ∞ as $x \to \mathbb{R}\backslash K$. For the proof to work in this more general formulation, we need

(i) w to be positive and continuous in a neighbourhood (in K) of each interpolation point;

(ii) the polynomials to be dense in a weighted Banach space of continuous functions.

Thus, one could assume, for example, that w is positive and continuous in the interior K° of K and that each $x_{jn} \in K^\circ$. Moreover, one can assume that the polynomials are dense in

$$C(w) := \{ f\colon K \to \mathbb{R} \text{ s.t. } f \text{ is continuous in } K^\circ \text{ and } \|fw\|_{L_\infty(K)} < \infty \}$$

and that

$$\|v/w\|_{L_p(K)} < \infty.$$

(The density is not trivial, and need not be true if $w(x) \to \infty$ fast enough as $x \to \mathbb{R}\backslash K$). If one wants only boundedness, and not convergence of $\{L_n\}$, then one can weaken these requirements on w.

We turn to:

Proof of Theorem 1. We let $C(K)$ denote the Banach space of continuous $f\colon K \to \mathbb{R}$ with norm

$$\|f\| := \|f\|_{L_\infty(K)}.$$

We suppose, as we may, that $K \subset [-1, 1]$.

(II) $\Rightarrow$ (I). We first suppose that $\|f\|_{L_\infty(K)} \leqslant 1$. Now we can write

$$L_n[f](x) = \pi_n(x) \sum_{j=1}^{n} \frac{f(x_{jn})}{\pi_n'(x_{jn})(x - x_{jn})} =: \pi_n(x)\, g_n(x).$$

Let $p > 0$. Then

$$\|L_n[f]\, v\|_{L_p(K)} \leqslant \|\pi_n v\|_{L_{2p}(K)} \, \|g_n\|_{L_{2p}(K)}. \tag{4}$$

To estimate the norm of g_n, we use its distribution function

$$m_{g_n}(\lambda) := meas\{x \in K : |g_n(x)| > \lambda\}, \qquad \lambda > 0.$$

Here *meas* denotes linear Lebesgue measure. A well known lemma of Loomis, that is often used in proving boundedness of the Hilbert transform between appropriate spaces (see [1, pp. 127–129; 2, p. 402, Lemma 3]) implies that

$$m_{g_n}(\lambda) \leqslant \frac{8}{\lambda} \sum_{j=1}^{n} \left| \frac{f}{\pi_n'}(x_{jn}) \right| \leqslant \frac{8}{\lambda} \sum_{j=1}^{n} \frac{1}{|\pi_n'|(x_{jn})} =: \frac{8}{\lambda} \Omega_n, \qquad \lambda > 0.$$

Moreover, there is the trivial bound $m_{g_n}(\lambda) \leqslant 2$ (the linear measure of $[-1, 1] \supseteq K$). We now use the representation of an L_p norm in terms of distribution functions [1, p. 43],

$$\begin{aligned} \|g_n\|_{L_{2p}(K)}^{2p} &= 2p \int_0^\infty \lambda^{2p-1} m_{g_n}(\lambda)\, d\lambda \\ &\leqslant 2p \int_0^\infty \lambda^{2p-1} \min\left\{2, \frac{8\Omega_n}{\lambda}\right\} d\lambda \\ &= 2p\Omega_n^{2p} \int_0^\infty s^{2p-1} \min\left\{2, \frac{8}{s}\right\} ds =: C_p^p \Omega_n^{2p}. \end{aligned}$$

Of course C_p is finite if $p < \frac{1}{2}$, which we now assume. (We note that the last estimate is essentially an inequality relating the weak L_1 norm of g_n and its L_{2p} norm.) Then (4) gives

$$\sup_n \|L_n[f]\, v\|_{L_p(K)} \leqslant C_p \sup_n \|\pi_n v\|_{L_{2p}(K)}\, \Omega_n < \infty,$$

by (2), provided $2p \leqslant r$. It then follows that for every $f \in C(K)$,

$$\sup_n \|L_n[f]\, v\|_{L_p(K)} \leqslant c\, \|f\|_{L_\infty(K)},$$

where c is independent of f. Next, let $\varepsilon > 0$. We may find a polynomial P such that

$$\|f - P\|_{L_\infty(K)} < \varepsilon.$$

Indeed, f has a continuous extension from K to $[-1, 1]$ and then Weierstrass' Theorem may be applied. Then for large enough n,

$$\begin{aligned} \|(f - L_n[f])\, v\|_{L_p(K)}^p &\leqslant \|(f - P)\, v\|_{L_p(K)}^p + (c\, \|f - P\|_{L_\infty(K)})^p \\ &\leqslant \varepsilon^p [\, \|v\|_{L_p(K)}^p + c^p], \end{aligned}$$

provided $p \leqslant q$, so that $\|v\|_{L_p(K)}$ is finite. Then the convergence (1) follows.

(I) ⇒ (II). We follow Shi [9, pp. 30–31, Lemma 1]. Assume that we have the convergence (1). Then the uniform boundedness principle gives

$$\|(f - L_n[f])\, v\|_{L_p(K)} \leqslant C \, \|f\|_{L_\infty(K)},$$

where C is independent of n and f, and consequently, for some possibly different C,

$$\|L_n[f]\, v\|_{L_p(K)} \leqslant C(\|f\|_{L_\infty(K)} + \|fv\|_{L_p(K)}). \tag{5}$$

Of course if $p < 1$, the space

$$\{h\colon K \to \mathbb{R} \text{ with } \|hv\|_{L_p(K)} < \infty\}$$

is not a normed space, but it is a topological vector space, while $C(K)$ is a Banach space, and there is a version of the uniform boundedness principle that may be applied. See, for example, [8, p. 44, Theorem 2.6]. Next, choose f continuous on K such that

$$f(x_{kn}) = sign(\pi_n'(x_{kn})), \qquad 1 \leqslant k \leqslant n$$

and $\|f\|_{L_\infty(K)} = 1$ (for example, we could choose f to be a piecewise linear function). We may also assume that the support of f is so small that

$$\|fv\|_{L_p(K)} \leqslant 1. \tag{6}$$

Let $S_n(x)$ be given by (3) and let $\sigma_n(x) := sign(\pi_n(x))$. We see that

$$\begin{aligned} S_n(x) &= \sigma_n(x)\, \pi_n(x) \sum_{k=1}^{n} \frac{f(x_{kn})}{\pi_n'(x_{kn})} = \sigma_n(x) \sum_{k=1}^{n} f(x_{kn})(x - x_{kn})\, \ell_{kn}(x) \\ &= \sigma_n(x)(xL_n[f](x) - L_n[g](x)), \end{aligned}$$

where $g(x) := xf(x)$. Then (5) and (6) and the fact that $|g| \leqslant |f|$ give

$$\begin{aligned} \|S_n v\|_{L_p(K)} &\leqslant 2^{1/p}(\|L_n[f]\, v\|_{L_p(K)} + \|L_n[g]\, v\|_{L_p(K)}) \\ &\leqslant 2^{1/p} C(\|f\|_{L_\infty(K)} + \|g\|_{L_\infty(K)} + 1) \leqslant 2^{1/p} 3C. \end{aligned}$$

As C is independent of n, we have (2) with $r = p$. ∎

ACKNOWLEDGMENT

The author thanks Paul Nevai for helping to improve the formulation of the result of this manuscript.

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